IV. Properties of Cluster Algebras

We keep saying cluster variables are "nice"
- what does this mean?

<u>Recall</u>: A Laurent polynomial is $\dfrac{\text{polynomial}}{\text{monomial}}$

<u>The Laurent Phenomenon</u> [Fomin-Zelevinsky '02]:
Every cluster variable in a cluster algebra $\mathcal{A}$ is a Laurent polynomial in the elements of any cluster. Hence

$$\mathcal{A} \subseteq \bigcap_{\substack{\underline{x} = (x_1, \dots, x_n) \\ \text{cluster}}} \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

<u>Note</u>: The RHS is called the <u>upper cluster algebra</u>, often it is equal to the cluster algebra

<u>Rmk</u>: This is surprising because we often divide by polynomials when mutating! The proof is not too complicated (see FWZ Ch 3.3), but we won't cover it here.

<u>Cor</u>: If we set all initial cluster variables equal to 1, then all cluster variables evaluate to integers

<u>Ex</u>: Markov numbers, friezes from worksheet

Laurent Positivity [Lee-Schiffler '15, Gross-Hacking-Keel-Kontsevich '18]

The coefficients of the Laurent expansion of a cluster variable in any cluster are positive integers.

Rmk: This is also very special, as we can easily get negative coeffs in polynomial division e.g. $\frac{x^3+1}{x+1} = x^2 - x + 1$.

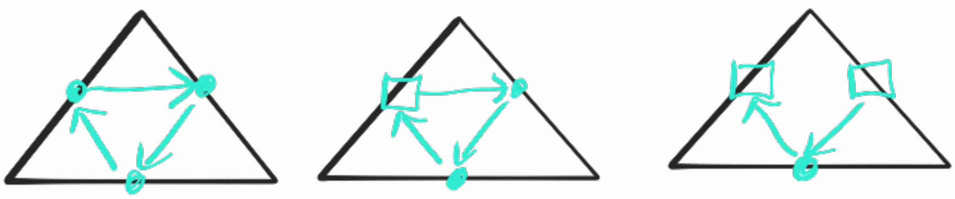
These 2 proofs are very deep!

Ex: Same as before, but now get positive integers :)

V. Cluster algebras from polygons

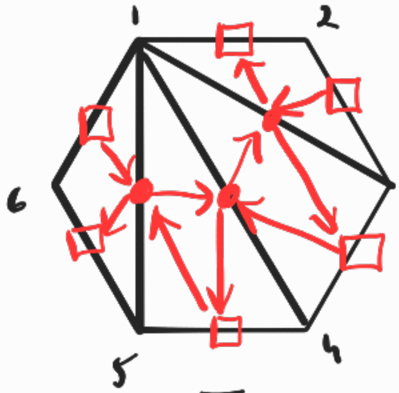
We can construct all seeds in the type A_d cluster algebra using triangulations of a $(n+3)$-gon.

Defn: Given a triangulation T of a $(n+3)$-gon, construct a quiver $Q(T)$ by putting a mutable vertex at each diagonal, and a frozen at each side. Add clockwise arrows within each triangle.

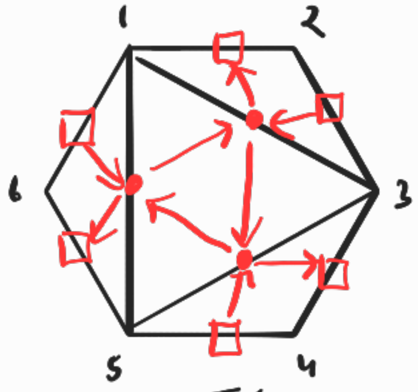


* ignore arrows between frozens

Example:

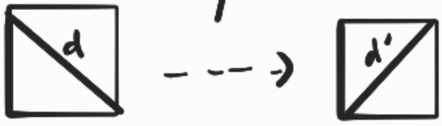


T



T'

Prop: Given a quadrilateral in T, we can perform a diagonal flip



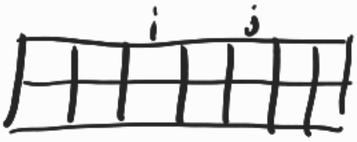
T T'

Then $Q(T') = \mu_d(Q(T))$.

Example: Q(T) and Q(T') differ only by a single mutation at the vertex inside the quadrilateral 1345

Note: If we take the "fan triangulation" where all diagonals meet at a vertex, the mutable part of the quiver is $\underbrace{\bullet \to \bullet \to \bullet \cdots \bullet \to \bullet}_{n}$

Grassmannian Interlude



Recall: Every full-rank $2 \times n$ matrix yields a point in $Gr(2,n)$

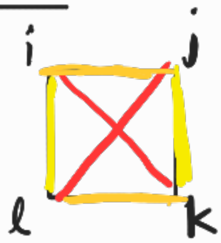
Defn: The Plücker coordinate P_{ij} of such a matrix is the 2×2 minor of the ith & jth columns

Fact: We have the Plücker embedding

$$\varphi: Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}}$$
$$M \longmapsto [P_{ij}]_{1 \le i < j \le n}$$

Lem: These satisfy the Plücker relation

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$$



Defn: The Plücker ring $R_{2,m}$ is
the ring generated by the P_{ij}
(only rels are Plücker relations)

We are often interested in the "positive part" of a space, where some important functions are positive

The totally positive Grassmannian
$Gr^+(2,n) \subset Gr(2,n)$ consists of points
that can be represented by a matrix
with all $P_{ij} > 0$

Problem: To check if rowspan$(M) \in Gr^+(2,n)$ by brute force, we'd need to compute $\binom{n}{2}$ minors.
Can we do fewer?

Back to cluster algebras!

Label the edge $\overline{ij}$ by P_{ij}, have $2n+3$ edges.

Lem: Diagonal flips $\longleftrightarrow$ Plücker

So the cluster & frozen variables really can be viewed as $P_{ij} \in Gr_{2,n+3}$

Using this, positivity of some Plückers can imply positivity for others. We'll show each cluster in $\mathcal{A}(Q(T))$ is a minimal test of total positivity for $Gr^+_{2,n+3}$

Fix a triangulation T of an $(n+3)$-gon.
Let $P_T = \{P_{ij} : \overline{ij} \in T\}$.

<u>Lem</u>: $\mathcal{A}(Q(T)) = R_{2,n+3}$

<u>pf</u>: First, note mutation = diagonal flip = Plücker relation

Via diagonal flips, we obtain any triangulation T' from T. Thus $\mathcal{A}(Q(T')) = \mathcal{A}(Q(T))$.
So every Plücker $P_{k\ell}$ is contained in $\mathcal{A}(Q(T))$, but also all cluster variables are Plückers. □

<u>Thm</u>: Any Plücker coordinate $P_{k\ell}$ can be written as a subtraction-free rational expression in the elements of P_T.

<u>pf</u>: By Laurent positivity, $P_{k\ell}$ is a Laurent polynomial in the initial cluster with coefficients in $\mathbb{Z}_{\geq 0}$ (hence subtraction-free) □

<u>Cor</u>: Any cluster in $\mathcal{A}(Q(T))$ (the type A_n cluster algebra) is a minimal test for total positivity in $Gr^+_{2,n+3}$

<u>Rmk:</u> This works for general TNN Grassmannians $Gr_{k,n}^+$ as well [Scott '06], and is highly related to Postnikov's plabic graphs & positroids